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Random Correspondences and Nonlinear Equations

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The question of existence of random solutions of nonlinear equations involving random correspondences is studied by direct use of the results from the deterministic case without resorting to iterative techniques. Applications to stochastic control and nonlinear random operator equations are given.

1. INTRODUCTION

In this paper we consider the question of existence of random solutions of nonlinear operator equations involving (possibly) multivalued random operators. Our aim is to point out that by using the known results and the structure of the solutions of the deterministic problems we can establish the randomness of the solution without having to look for iterative techniques.

Thus let $(\Omega, \mathscr{B}, \mu)$ be a complete probability space and X, Y be two Banach spaces. We consider a nonlinear random operator equation of the type $T(\omega, x) = y(\omega)$ for the existence of random solutions where $y(\omega)$ is a given Y-valued random variable and $T\colon \Omega \times X \to Y$ is a random nonlinear operator. The general procedure for establishing the existence of a random solution is as follows: for each $\omega \in \Omega$, we solve the corresponding deterministic problem and obtain the collection of all solutions $x \in X$. The measurability of the map which associates to each $\omega \in \Omega$ the set of all solutions of the deterministic problem is then established.

If there exists an iterative technique to solve the deterministic problem, then by virtue of the randomness of T and the principle of superposition the solution of the random problem can be shown to be random. However if the deterministic problem is not solved iteratively, then one has to resort to other techniques. For a brief survey of some of the results using the iterative technique one is referred to [2]. In [13, 14] we show that by applying a result due to Debreu [8] on the equivalence of the measurability of the graph

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of a random operator and the measurability of the operator we can establish the randomness of the solution for single-valued operator equations. In this paper we propose to study the case of nonlinear random problems involving multivalued random operators, thereby answering some questions raised by Bharucha-Reid [3]. Recently Sussman [26] points out that stochastic differential equations involving Wiener processes can be handled by the general technique of the preceding paragraph. This fact combined with our results for multivalued random problems should enable us to handle multivalued analogues of the recent work on stochastic differential equations involving monotone operators, thereby enabling us to handle the case of subdifferentials.

Thus in order to handle random correspondences we want to obtain sufficient conditions for the randomness of T^{-1} (possibly multivalued). This together with the hypotheses that the deterministic problem is solvable enables us to obtain the randomness of $T^{-1}y$, the solution of $T(\omega, x) = y(\omega)$. Hans [10] was the first to study rigorously the randomness of the inverse of a one-to-one and bounded linear operator. In [20] Nashed and Salehi extend this study to obtain sufficient conditions for the measurability of the generalized inverse of a separable single-valued random operator. In this paper we obtain a multivalued analogue of their theorem and show in the final section that the sufficiency hypotheses of our theorem are satisfied in a wide variety of situations, involving random operator equations. Using known results on existence of measurable selections we can obtain a measurable single-valued solution to the random operator equation. We are now in a setting to study the statistical properties and distribution of the solution process $x(t, w)$. Often, in the deterministic case, we consider a sequence of nonlinear problems whose solutions converge (in some sense) to the solution of the given nonlinear problem. Since the techniques used in this paper to establish the randomness of the solution utilize the properties of the deterministic problem, we are able to obtain estimates on the moments of the solution of the approximate problem immediately. These are then used to obtain limit theorems for the given random nonlinear problem. Details of these may be seen in our forthcoming paper [5].

Another application of the ideas in this paper is to the area of stochastic optimal control. In Section 3 we obtain a multivalued analogue of the McShane–Warfield generalization of Filippov's implicit function theorem. We then show how this can be applied to establish the existence of measurable optimal controls.

A detailed survey on some aspects of random operator equations may be seen in [11].

2. Preliminaries

We now recall some definitions and known results from the theory of random operator equations. Let $(\Omega, \mathscr{B}, \mu)$ be a complete probability space, X a separable Banach space, and Y a Banach space. The mapping T from $\Omega \times X$ into Y is said to be a *random operator* if for each fixed $x \in X$, $T(\cdot, x)$ is a random variable. For a detailed study of random operators we refer to [25]. A correspondence T of Ω into B is a mapping of Ω into the set of all nonempty subsets of B. Let $\mathscr{B}$ be a σ-algebra on Ω and let ρ be a metric on B. A correspondence T is called $\mathscr{B}$-measurable if for any $x \in B$ the function $\rho(x, T(\omega))$ is $\mathscr{B}$-measurable. A closed-valued correspondence $T: \Omega \to X$ is said to be random if $T^{-1}(B)$ is measurable for each closed subset B of X. We now have the following:

Proposition 1. *For a closed-valued correspondence $T: \Omega \to X$ the following are equivalent*: (a) *T is random*; (b) *$T^{-1}(B)$ is measurable for all open sets B*; (c) *$T^{-1}(B)$ is measurable for all compact sets B*; (d) $\operatorname{dist}(z, T(\omega))$ *is a measurable function of ω for each $z \in X$*; (e) Graph *T is $\mathscr{B} \times B_x$ measurable.*

Proposition 2 ([16]). *A closed-valued random correspondence $T: \Omega \to X$ has a measurable selection.*

A single-valued random operator $T: \Omega \times X \to Y$ is said to be *continuous* at $x_0 \in X$ if $\lim_{n \to \infty} \| x_n - x_0 \| = 0$ implies $\lim_{n \to \infty} \| T(\omega, x_n) - T(\omega, x_0)\| = 0$ a.s. The random operator $T: \Omega \times X \to Y$ is said to be *separable* if there exists a countable set $S \subset X$ and a negligible set $N \in \mathscr{B}$, $\mu(N) = 0$ such that

$$\{\omega: T(\omega, x) \in K, x \in F \cap S\} \triangle \{\omega: T(\omega, x) \in K, x \in F\} \subset N$$

for every closed set K in $\mathscr{B}_Y$ and every F in $\mathscr{B}_X$. An equation of the type $T[\cdot, x(\cdot)] = y(\cdot)$ where y is a given random variable with values in Y is called a *random operator equation.* Any X-valued random variable which satisfies $\mu\{\omega: T(\omega, x(\omega)) = y(\omega)\} = 1$ is said to be a *random solution* of the above random operator equation. For a single-valued random operator we have the following:

Proposition 3 [4]. *Any continuous random operator $T: \Omega \times X \to X$ is separable.*

Proposition 4 [4]. *Let $T: \Omega \times X \to E$ be a separable random operator where E is a compact subset of X. Then $\| T(\omega)\|$ is a nonnegative real-valued random variable.*

PROPOSITION 5 [10]. *Let $T: \Omega \times X \to Y$ be a separable random linear operator such that a.s. $T(\omega, \cdot)$ is invertible and $T^{-1}(\omega, \cdot)$ is bounded. Then T^{-1} is a random operator from $\Omega \times Y$ into X.*

PROPOSITION 6 [18]. *Let Ω be a measure space, Y a Hausdorff space, and X a topological space which is the union of a countable number of compact metrizable sets. Let $k: X \to Y$ be continuous and $y: \Omega \to Y$ a measurable function such that $y(\Omega) \subseteq k(X)$. Then there exists a measurable function $u: \Omega \to X$ such that*

$$k(u(\omega)) = y(\omega) \qquad \text{for all } \omega \text{ in } \Omega.$$

We conclude this section with the following definition of upper-semicontinuity. A multivalued operator $T: X \to Y$ is said to be *upper-semicontinuous* if (i) for each $x \in X$, Tx is closed and (ii) for any sequence $\{x_n\} \to x$ in X and $y_n \to y$ in Y where $y_n \in Tx_n$, we have $y \in T(x)$. In the rest of this paper we will assume that T is convex valued.

3. RANDOM CORRESPONDENCES

Before we obtain sufficient conditions for the randomness of the inverse of a random correspondence we first obtain some multivalued analogues of Propositions 3 and 4. Let X be a separable Banach space.

THEOREM 1. *Let $T: \Omega \times X \to X$ be a compact-valued random correspondence such that for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper-semicontinuous and for each $x \in X$, $T(\cdot, x)$ is random. Then T is $\mathscr{B} \times \mathscr{B}_X$ measurable where $\mathscr{B} \times \mathscr{B}_X$ is the product σ-algebra on $\Omega \times X$.*

Proof. The proof follows by a standard construction in measure theory (cf. [14, 21]). Thus we approximate the correspondence T by a sequence of correspondences T_i each of which is $\mathscr{B} \times \mathscr{B}_X$ measurable. Proceeding as in [14] we define a sequence of maps $T_j: \Omega \times X \to X$ as follows. For each j let C_{ij} be a sequence of closed sets in $\mathscr{B}_X$ generated by a countable everywhere dense set $\{x_n\}$ in X such that $\bigcup_i C_{ij} = X$ and $C_{ij} \cap C_{kj} = \Phi$ if $i \neq k$. Let

$$T_j(\omega, x) = T(\omega, x_i) \qquad \text{for} \quad \omega \in \Omega \quad \text{and} \quad x \in C_{ij}.$$

For any open subset C of X,

$$\{(\omega, x): T_j(\omega, x) \cap C \neq \Phi\} = \bigcup_{i=1}^{\infty} [\{\omega \in \Omega: T(\omega, x_i) \cap C \neq \Phi\} \times C_{ij}].$$

But the latter is in $\mathscr{B} \times \mathscr{B}_X$ and thus T_j is $\mathscr{B} \times \mathscr{B}_X$ measurable for each j. Hence, from the previous section, $\operatorname{dist}(x, T_j(\omega))$ is measurable in ω for each $x \in X$. By the assumptions on T it follows that $\operatorname{dist}(z, T_j(\omega)) \to \operatorname{dist}(z, T(\omega))$ as $j \to \infty$ and hence T is $\mathscr{B} \times \mathscr{B}_X$ measurable.

Remark 1. The analogous version of the above for single-valued T may be seen in [14]. Our proof here follows this on identical lines.

COROLLARY 1. *A superposition principle that follows from the above theorem is that if $\phi: \Omega \to X$ is random, then the map $N: \Omega \times X \to X$ defined by $N(\omega) = T(\omega, \phi(\omega))$ is also a random correspondence.*

In applications to differential and integral equations, we often face the situation when a multivalued nonlinear operator T maps $\Omega \times X$ into X^*, where X is a separable reflexive Banach space. For such situations we can obtain the following interesting result regarding the randomness of T. A nonlinear operator $T: X \to X^*$ is said to be *demicontinuous* if $u_n \in D(T)$ (the domain of T) and $u_n \to u$ in X implies $Nu_n \to Nu$. Demicontinuity plays an important role in the theory of nonlinear maximal monotone operators.

THEOREM 2 [15]. *Let $(\Omega, \mathscr{B}, \mu)$ be a complete probability space, and X and Y separable Banach spaces. Let $T: \Omega \times X \to Y$ be a random correspondence such that for each $\omega \in \Omega$, T is demicontinuous. Then for any X-valued random variable $x(\omega)$, $T(\omega, x(\omega))$ is a Y-valued random correspondence.*

Proof. Consider the multivalued function $f: \Omega \times X \to R$ defined by $f(\omega, x) = \{\langle y^*, T(\omega, x)\rangle\}$ where y^* is any given element of Y^*. Clearly f is measurable in ω. Also the demicontinuity of T implies that f is upper-semicontinuous in X. Hence, by Corollary 2, it follows that $f(\omega, x(\omega))$ is a random correspondence. But this is true for each $y^* \in Y^*$. Hence $T(\omega, x(\omega))$ is a random correspondence because of the equivalence of weak and strong measurability when X and Y are separable [2, p. 15].

Proceeding analogously to Corollary 1 and Theorem 2 and applying a theorem due to Castaing [6] we have:

COROLLARY 2. *Let $T: \Omega \times X \to X$ be a closed-valued random correspondence and let $f: \Omega \times X \to Y$ be a function which is measurable in ω for each $x \in X$ and continuous in x for each $\omega \in \Omega$, where X and Y are separable metric spaces. Then the correspondence defined by*

$$\phi(\omega) = \{x \in T(\omega): f(\omega, x) \in \mathscr{U}\},$$

where $\mathscr{U}$ is a open set in Y, is random.

As a generalization of Proposition 3 we have:

THEOREM 3. *Let $T: \Omega \times X \to X$ be a.s. an upper-semicontinuous compact-valued random correspondence where X is a separable Banach space. Then T is separable.*

Proof. Let $\mathscr{F}$ be the countable class of all sets of the form

$$\{x: \|x - x_i\| \leqslant r\}, \qquad \{x: \|x - x_i\| \geqslant r\}$$

and their finite intersections where $\{x_i\}$ is dense in X and r is a rational number.

For any closed subset F in $\mathscr{F}$ let S_F be a countable everywhere dense set in F and let $S = \bigcup_{F \in \mathscr{F}} S_F$.

Let N be a negligible set and let $\omega \notin N$ be such that $T(\omega)$ is upper-semicontinuous from X into X. Let x be any element of X and $z \in T(\omega)x$. Now the set $\{y \in X: \|y - z\| < \varepsilon\}$ has nonempty intersection with $T(\omega)(F \cap S)$. This follows from the fact that since T is upper-semicontinuous, we can find a ball around x such that the image of this ball is contained in the above open set. Since S_F if dense, the nonemptiness of the intersection with $T(\omega)(F \cap S)$ follows. Hence, for $\omega \notin N$,

$$T(\omega)(x) \in \bigcap_{\substack{x \in F \\ F \in \mathscr{F}}} \overline{T(\omega)(F \cap S)}.$$

Thus for $\omega \notin N$, $\overline{T(\omega)(F)} = \overline{T(\omega)(F \cap S)}$ and hence T is separable.

We now obtain the multivalued analogue of the theorem of Nashed and Salehi [20].

THEOREM 4. *Let $(\Omega, \mathscr{B}, \mu)$ be a complete probability space, and X and Y separable Banach spaces. Let $T: \Omega \times X \to Y$ be a separable random correspondence such that a.s. $T(\omega, \cdot)$ is invertible and $T^{-1}(\omega, \cdot)$ is upper-semicontinuous and compact valued. Then T^{-1} is also a random correspondence from $\Omega \times Y$ into X.*

Proof. It suffices to show, by Proposition 1, that for any $y \in Y$ and a closed ball $\bar{S}(x', r)$, the event $\{\omega: T^{-1}(\omega, y) \in \bar{S}(x', r)\} \in \mathscr{B}$. But $\{\omega: T^{-1}(\omega, y) \in \bar{S}(x', r)\} = \bigcup_{x \in \bar{S}(x', r)} \{\omega: y \in T(\omega, x)\}$. We now show that

$$\bigcup_{x \in \bar{S}(x', r)} \{\omega: y \in T(\omega, x)\}$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{x \in S(x', r + 1/n)} \{\omega: \operatorname{dist}[y, T(\omega, x)] < 1/n\}.$$

Clearly the left-hand side is contained in the right-hand side. Now let ω_0 belong to the right-hand side. Then, for each n, there exists $x_n \in$

$S(x', r+1/n)$ such that $\text{dist}[y, T(\omega_0, x_n)] \leqslant 1/n$. Hence there exist $z_n \in T(\omega_0, x_n)$ such that $z_n \to y$. But $x_n \in T^{-1}(z_n)$ and the x_n are bounded. Hence by the closed graph continuity of T^{-1}, if x_n (or a subsequence) $\to x$, then $x \in T^{-1}(\omega_0, y)$ or $y \in T(\omega_0, x)$. Also $x \in \bar{S}(x', r)$. Thus $\omega_0 \in \bigcup_{x \in \bar{S}(x', r)} \{\omega: y \in T(\omega, x)\}$.

Finally we note that

$$\left[\bigcup_{n=1}^{\infty} \bigcup_{x \in S(x', r+1/n)} \{\omega: \text{dist}(y, T(\omega, x)) < 1/n\}\right]^c$$

$$= \bigcup_{n=1}^{\infty} \bigcap_{x \in S(x', r+1/n)} \{\omega: \text{dist}(y, T(\omega, x)) \geqslant 1/n\}.$$

Since T is separable, it follows that the right-hand side is measurable. This proves the theorem.

We conclude this section with a generalization of a result due to McShane and Warfield. They generalized a result, due to Filippov [9]. The idea in Filippov's result is as follows: let k be a continuous function on a compact set X in a finite-dimensional space with values in a finite-dimensional space and let $(u'(t): a \leqslant t \leqslant b)$ be a function with values in X such that $k(u'(t))$ is measurable; then there exists a measurable function u from $[a, b]$ to X such that $k(u(t)) = k(u'(t))$. Keeping in mind applications to stochastic optimal control theory, McShane and Warfield generalized Filippov's result to arbitrary measure spaces and this was stated in the previous section. We now obtain a multivalued analogue of their result, the proof of which follows their proof very closely and hence is only sketched here.

THEOREM 5. *Let $(\Omega, \mathscr{B}, \mu)$ be a complete probability space, Y a separable metric space, and X a union of countably many compact metrizable subsets. Let $T: \Omega \times X \to Y$ be a compact-valued random correspondence such that for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper-semicontinuous and for each $x \in X$, $T(\cdot, x)$ is random. Let $y: \Omega \to Y$ be random such that*

$$y(\omega) \in T(\omega, X).$$

Then there exists a random $u: \Omega \to X$ such that

$$y(\omega) \in T(\omega, u(\omega)).$$

Proof. Since the proof is similar to that of McShane and Warfield we use the same notations as far as possible. As in their proof it suffices to consider the case when X is a closed subset of $(0, \infty)$ which we denote by L. We then consider the sets

$$B_j^q = \{\omega: y(\omega) \in T(\omega, L \cap [0, j \cdot 2^{-q}]) - T(\omega, L \cap [0, (j-1) \cdot 2^{-q}]\},$$

where q is a nonnegative integer and $j = 1, 2, \dots$. We must first show that $B_j^q \in \mathscr{B}$. We prove this for any compact set $\mathscr{B}$, i.e., we show

$$\{\omega: y(\omega) \in T(\omega, L \cap B)\} \in \mathscr{B}.$$

As in the proof of Theorem 4, we first prove that

$$\begin{aligned}&\{\omega: y(\omega) \in T(\omega, L \cap B)\}\\&\quad = \bigcup_n \bigcap_{C \in P_n} \{\omega: y(\omega) \in C \text{ and } T(\omega, L \cap B) \cap C \neq \Phi\},\end{aligned}$$

where P_n is a countable cover of Y by open sets of diameter ε_n, ε_n being a decreasing sequence of positive numbers converging to zero. The proof uses the compact upper-semicontinuity (u.s.c.) of T for each ω. We then show, once again using the compact u.s.c. of T and separability of $L \cap B$ that for each open set C, $\{\omega: T(\omega, L \cap B) \cap C \neq \Phi\} \in \mathscr{B}$ and by hypothesis $\{\omega: y(\omega) \in C\} \in \mathscr{B}$. Hence

$$\{\omega: y(\omega) \in T(\omega, L \cap B)\} \in \mathscr{B}.$$

We now define the random functions $h_q(\omega)$ by

$$h_q(\omega) = \inf\{h: h \in L \cap ((j-1) \cdot 2^{-q}, j \cdot 2^{-q})\},$$

where j is that integer such that $y(\omega) \in B_j^q$. As in McShane and Warfield [18], it can be shown that $h_q(\omega)$ is an increasing sequence of bounded random functions. In fact

$$h_0(\omega) = \inf\{h: h \in L \cap (j-1, j)\}$$

and $h_1(\omega) = \inf\{h: h \in L \cap ((i-1)/2, i/2)\}$, where $y(\omega) \in L \cap (j-1, j)$ and $y(\omega) \in L \cap ((i-1)/2, i/2)$. Thus $i = 2j$ or $2j-1$ and in either case by definition it follows that

$$h_1(\omega) = \inf\left\{h: h \in L \cap \left(\frac{2j-1}{2}, \frac{2j}{2}\right)\right\}$$

or

$$h_1(\omega) = \inf\left\{h: h \in L \cap \left(\frac{2j-2}{2}, \frac{2j-1}{2}\right)\right\}.$$

Thus $h_1(\omega) \geqslant h_0(\omega)$ and this process can be continued. It is also seen from above that $h_0(\omega) \leqslant j$ and $h_1(\omega) \leqslant j$. Extending this argument, we see that

$h_q(\omega)$ is bounded. Hence $h_q(\omega)$ converges to a random function $h(\omega)$. It remains to show that

$$y(\omega) = T(\omega, h(\omega)).$$

If not, by the upper-semicontinuity of T for each ω, there exists a $\omega \in \Omega$ and j, q such that $y(\omega) \notin \mathscr{U}$,

$$h(\omega) \in L \cap ((j-1) \cdot 2^{-q}, j \cdot 2^{-q})$$

and

$$L \cap ((j-1) \cdot 2^{-q}, j \cdot 2^{-q}) \subseteq [T(\omega, \cdot)]^{-1}\mathscr{U},$$

$\mathscr{U}$ being a neighborhood of $T(\omega, h(\omega))$.

Now $h_q(\omega) \in L$ and $h_q(\omega) \leqslant h(\omega) \leqslant j \cdot 2^{-q}$. Hence $h_m(\omega) \leqslant 2^{m-q}(j-1)\, 2^{-m}$ by induction for $m \geqslant q$. Thus $h_m(\omega) \leqslant (j-1)\, 2^{-q}$ and thus $h(\omega) \leqslant (j-1)\, 2^{-q}$ which is a contradiction.

Hence $y(\omega) = T(\omega, h(\omega))$ and this proves the theorem for the case $X = L$. The general case can be proved as in [18].

4. Application to Random Operator Equations

In this section we will give some applications of the results of the preceding section to random nonlinear problems thereby illustrating how one proceeds to establish the randomness of the solution. We do not try to be exhaustive, covering most of the cases for which the deterministic theory is well developed but our emphasis is on giving sufficient examples to outline the general procedure.

Nonexpansive maps. A Banach space X is said to satisfy Opial's condition if for any sequence x_n converging weakly to x in X we have $\liminf \|x_n - y\| > \liminf \|x_n - x\|$ for $y \neq x$ (cf. [22]).

THEOREM 6. *Let $(\Omega, \mathscr{B}, \mu)$ be a complete probability space and let X be a separable Banach space which satisfies Opial's condition. Let C be a nonempty weakly compact convex subset of X and $T\colon \Omega \times C \to C$ be a random correspondence such that T is compact valued and*

$$D(T(\omega, x), T(\omega, y)) \leqslant \|x - y\| \quad \text{for} \quad \omega \in \Omega \quad \text{and} \quad x, y \in C,$$

where D denotes the Hausdorff metric. Then there exists $x_0(\omega)$ such that

$$x_0(\omega) \in T(\omega, x_0).$$

Proof. Since T is compact valued for each ω and T is nonexpansive, T satisfies the hypotheses of Theorem 3. Hence T is separable. It remains to verify the hypotheses of Theorem 4. For each $\omega \in \Omega$, the operator T satisfies the hypotheses of Lami Dozo [17] and hence there exists x_0 such that $x_0 \in T(\omega, x_0)$ for each fixed $\omega \in \Omega$. We now show that $(I-T)^{-1}$ is closed-graph continuous, i.e., given that $y_n \to y$ and $x_n \to x$ where $x_n \in (I-T)^{-1} y_n$ we need to show that $y \in (I-T)x$. Clearly $x_n \in C$ implies that $x \in C$. Also $y_n \in (I-T)\, x_n$ implies $y_n = x_n - v_n$, $v_n \in Tx_n$. Since T is nonexpansive with respect to the Hausdorff metric we can find $v'_n \in Tx$ such that

$$\|v_n - v'_n\| \leqslant \|x_n - x\|.$$

Thus

$$\liminf \|x_n - x\| \geqslant \liminf \|v_n - v'_n\| \geqslant \liminf \|x_n - y_n - v'_n\|.$$

Since T is compact valued and $y_n \to y$, passing to subsequences we have $v'_n \to v \in Tx$ and thus

$$\liminf \|x_n - x\| \geqslant \liminf \|x_n - y - v\|.$$

By Opial's condition it follows that $y + v = x$. Thus $y = x - v \in x - Tx$. Hence by Theorem 4, $(I-T)^{-1}$ is a random correspondence and thus there exists a random solution of the random operator equation $x(\omega) - T(\omega, x) = 0$.

Remarks. Hilbert spaces, l^p $(1 \leqslant p < \infty)$, finite-dimensional spaces, uniformly convex Banach spaces possessing a weakly continuous duality mapping are examples of Banach spaces satisfying Opial's condition.

The above method of proof can also be adapted to the case of a Banach space which is uniformly convex in every direction. Thus we have obtained a random multivalued analogue of the Browder–Gohde–Kirk fixed point theorem. Theorem 5 also generalizes to the random case the well-known results on point-to-set contraction mappings.

α-Condensing mappings. As our next application we shall obtain a random analogue of a fixed point theorem for condensing mappings. Let X be a Banach space and T be a map of X into itself. Then we say that T is α-condensing if for every $B \subset X$ the inequality $\alpha(TB) \geqslant \alpha(B)$ implies $\alpha(B) = 0$ where $\alpha(B) = \inf\{r\colon B$ can be covered by a finite number of sets of diameter less than $r\}$. α-condensing maps include as particular cases compact maps and compact perturbations of contraction maps. Then as corollaries we can find random analogues of the Schauder and Krasnoselskii fixed point theorems.

THEOREM 7. *Let $(\Omega, \mathscr{B}, \mu)$ be a complete probability space and let X be a separable Banach space. Let $T: \Omega \times G \to G$ be an α-condensing map where G is a bounded closed convex subset of X. Then there exists a random fixed point of T, i.e., there exists a random solution of the equation*

$$x(\omega) = T(\omega, x(\omega)).$$

Proof. For each fixed ω, it is known that under the hypotheses of the theorem, T has a fixed point. The separability of T is obvious by Theorem 3. We now show that $(I-T)^{-1}$ is closed-graph continuous. Thus let $y_n \to y$ and $x_n - Tx_n = y_n$. It must be noted that $(I-T)^{-1}$ is not necessarily single valued. Then

$$\alpha(x_n) = \alpha(Tx_n + y_n) \leqslant \alpha(Tx_n) = \alpha(y_n).$$

Since $y_n \to y$, $\alpha(y_n) = 0$. Also since T is α-condensing the above implies that x_n is precompact and the continuity of T implies that each limit point of x_n is a solution of $x - Tx = y$. Then $(I-T)^{-1}$ is closed-graph continuous and hence by Theorem 4 the existence of a random fixed point follows.

Note. As remarked at the beginning of this application the above theorem yields random analogues of the Schauder and Krasnoselskii theorem also and thus provides a direct proof of these theorems [3]. In particular by an application of this theorem we can prove the existence of a random solution to nonlinear integral equations of the type considered by Miller, Nohel, and Wong [19] thereby taking care of the difficulty in [23].

Maximal monotone operators. We now show that maximal monotone operators satisfy the hypotheses of Theorem 4 and thus the random analogue of nonlinear evolution equations involving monotonic nonlinearities can be obtained. This is of particular significance in considering stochastic differential equations of the Navier–Stokes type as considered by Bensoussan and Temam [1]. For, as has been established by Sussman [26], such problems can now be treated by the direct method. We recall that a mapping T of a real Banach space into its dual space X^* is monotone if $(y_1 - y_2, x_1 - x_2) \geqslant 0$ where $y_1 \in T(x_1)$ and $y_2 \in T(x_2)$. T is said to be maximal monotone if it is maximal in the sense of ordering by inclusion of graphs. We now obtain the following:

THEOREM 8. *Let $(\Omega, \mathscr{B}, \mu)$ be a complete probability space and let X be a separable Banach space. Let $T: \Omega \times X \to X^*$ be maximal monotone with Tx nonempty for each $x \in X$. Further let the equation $0 \in Tx$ have a solution for each $\omega \in \Omega$. Then there exists an X-valued random variable $x(\omega)$ such that*

$$0 \in Tx(\omega).$$

Proof. In order to apply Theorem 4, we need to verify just the closed-graph continuity of T^{-1} and the separability of T. Both of these follow from the theory of maximal monotone operators. Thus let $\{x_n\}$ converge to x_0 in X and let $y_n \in T(x_n)$. By a result due to Rockafellar [24] it follows that $\{y_n\}$ is bounded. By the separability of X, it suffices to show that if $\{y_n\}$ is weak*-convergent to y_0, then $y_0 \in T(x_0)$. Since T is monotone, $\langle y_n - b, x_n - a\rangle \geqslant 0$ for $a \in X$ and $b \in T(a)$. Proceeding to the limit $\langle y_0 - b, x_0 - a\rangle \geqslant 0$ for all $a \in X$, $b \in T(a)$. Since T is maximal monotone, it follows that $y_0 \in T(x_0)$. The closed-graph continuity of T implies that T is separable by Theorem 1. Also the inverse of a maximal monotone operator is also maximal monotone and hence the hypotheses of Theorem 4 are satisfied. Thus T^{-1} is also random and this completes the proof.

Remarks. In all of the above applications it can be seen that the map T^{-1} is closed valued for each ω. And since X is separable, it follows from the well-known selection theorem [6, 16] that there exists a single-valued random fixed point (or solution) in the above problems.

Stochastic optimization. We conclude the paper with another application of the ideas of Section 3 to a problem in stochastic optimization. Thus we consider the problem of maximizing the functional $I(\omega, x, u)$ subject to the set of constraints characterized by $0 \in \phi(\omega, x, u)$. Here we assume that $x(\omega)$ and $y(\omega)$ are in the ranges of random correspondences $A: \Omega \to 2^X$, $B: \Omega \to 2^U$, X and U being separable Banach spaces.

An example of the above would be, as in Cesari [7], to maximize $I(\omega, x, u) = \int_{t_1}^{t_2} f_0(\omega, t, x(t, \omega), u(t, \omega))\, dt$ with $dx/dt = f(\omega, t, x, u)$ and appropriate constraints on $u(t, \omega) \in U(\omega, t, x)$ together with the boundary conditions.

We assume for the rest of the discussion that the hypothesis on I and ϕ are such that, for any $\omega \in \Omega$, the deterministic problem is solvable. In order to prove the randomness of the correspondence which associates with each $\omega \in \Omega$ the set of couples (x, u) on which I attains its maximum subject to the constraints, we proceed in the following three steps:

(i) established the randomness of the correspondence T_1, which is defined by

$$T_1(\omega) = \{(x, u) : x \in A(\omega), u \in B(\omega) \text{ and } 0 \in \phi(\omega, x, u)\};$$

(ii) show the correspondence $g(\omega)$ given by

$$g(\omega) = \sup_{(x,u) \in T_1(\omega)} I(\omega, x, u)$$

is random;

(iii) finally show that the correspondence

$$T(\omega) = \{(x, u) \in T_1(\omega) \colon I \text{ attains its maximum at } (\omega, x, u)\}$$

is random.

All three steps are proved by using the results of Section 3 and [14], where we use Proposition 1(e) from Section 2. We omit the details here because the proof is a routine application of the continuity of I and the upper-semicontinuity of ϕ.

The above setting also arises in problems of decision theory where one has to show the existence of admissible controls y minimizing a cost functional $E[\int_a^b f(\omega, t, x(t, \omega), y(t, x)\, dt]$, where $x(t, \omega)$ is a continuous stochastic process. Appropriate hypotheses on the moments of x and assumptions on f related to the methods in [7] lead to establishing the existence of optimal $y(t, x)$.

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